DOMAIN

Have you ever gotten an ERROR message on your calculator? You may have tried to divide by zero — or perhaps you attempted to compute the square root of a negative number. In either case, you tried to use a number outside the *domain* of the *function* you were trying to calculate.

❑ TWO WAYS TO DESCRIBE THE DOMAIN

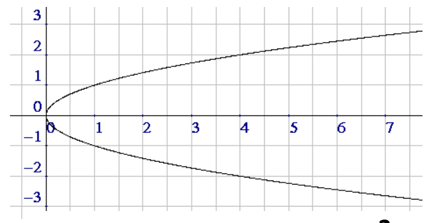
The *domain* of a function is the set of all inputs to the function.

There are two ways we specify the domain of a function.

Sometimes, the domain is *explicitly* given (very common in Calculus). For instance, let h be the function defined on the set of numbers [0, 3] by the formula $h(x) = x^2$. The domain of this function is the set [0, 3]. Why? Because the definition says so. Thus, while $h(0) = 0$ and $h(2.5) = 6.25$ and $h(3) = 9$, the fact is that $h(-1)$ is undefined and $h(4)$ is also undefined. It's not that –1 and 4 can't be squared — they simply are not in the explicitly given domain, [0, 3].

Other times, the domain is not explicitly given, so we agree to use the ***natural domain***). In this scenario, the function's domain is every real number that's legal to use in the formula. For example, the function $g(x) = x^2$ would have a domain of $\mathbb{R}$, because any real number can be squared. But in the formula $y = \frac{3}{x-7}$, x can be any real number <u>except</u> 7 (otherwise, we're dividing by 0). Therefore, the domain is **$\mathbb{R}$ – {7}**.

The concept of domain also applies to formulas that are not functions. Consider the formula $x = y^2$. We know this is not a function because an input of $x = 25$, for instance, produces two outputs, $y = \pm 5$. Nevertheless, we can still ask what inputs are allowed. Since x is the square of y, it should be clear that x must be greater than or equal to zero. That is, the domain is **[0, ∞)**.



The non-function $x = y^2$ has domain [0, ∞)

Homework

1. Consider the function $f(x) = \frac{6x-6}{\sqrt{2x-8}}$. Calculate each functional value: a. $f(12)$ b. $f(36)$ c. $f(4.5)$ d. $f(4)$ e. $f(3)$
From these results we see that 12, 36, and 4.5 are in the _______ of the function, while 4 and 3 are not.

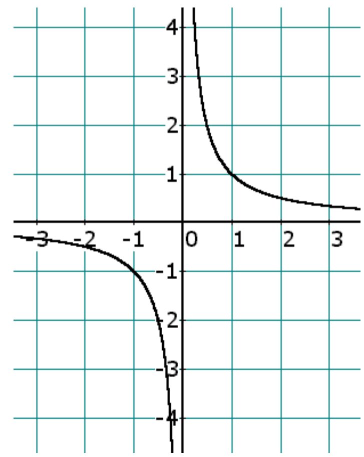
2. What is the domain of the function defined by $y = x^3$, $x \in [3, 5]$?

3. Consider the function given by the formula $g(x) = x^3 - x^2$. What is the domain of g?

❑ *PRELIMINARY EXAMPLES*

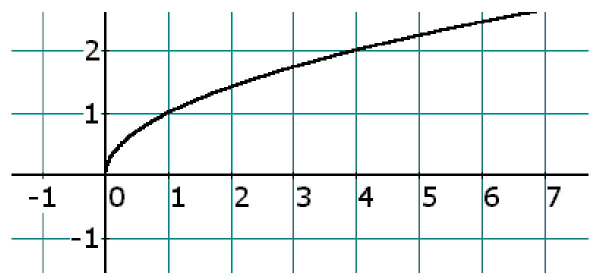
We have a pretty good notion of what a function is. There's a set of *inputs*, which are in some way associated with a set of *outputs*. Although this association is usually given by a formula (e.g., $y = x^2$), don't forget about the four quarters of the football game and the graphs in the previous chapter. Most importantly, remember that this association between the inputs and the outputs is such that *each element in the set of inputs is associated with exactly one output*. This chapter focuses on what inputs are allowed in a function.

First Example: Consider the function $y = \frac{1}{x}$. What inputs are allowed in this function? That is, what can x legally be? Well, x had better not be 0, since division by 0 is undefined. But if x is any other number, no problems will occur. So x can be any real number except 0.



The set "All real numbers except 0" can be written $\mathbb{R} - \{0\}$.

Second Example: Now consider the function $y = \sqrt{x}$. We ask the same question: What x's are allowed in this formula? In other words, what kinds of real numbers are we allowed to take the square root of? The answer is that we can take the square root of any number that is zero or greater — which is equivalent to saying that we cannot take the square root of a negative number. Thus, x can be any number greater than or equal to 0, which we also write as $\boldsymbol{x \geq 0}$, or, if you're familiar with interval notation: $\mathbf{[0, \infty)}$.



Homework

4. Let $y = 2x + 10$. Which of the following are legal values of x?

a. 30　　b. $-\pi$　　c. $\sqrt{7}$　　d. 0

5. Let $y = \frac{3}{x-2}$. Which of the following are legal values of x?

   a. 3 b. 2 c. π d. 0

6. Let $y = \sqrt{x}$. Which of the following are legal values of x?

   a. 7 b. 0 c. –9 d. 99

7. Let $y = \frac{10}{x^2 - 144}$. Which of the following are legal values of x?

   a. 12 b. 20 c. 0 d. –12

8. Let $y = \frac{x}{x^2 + 9}$. Which of the following are legal values of x?

   a. 3 b. 0 c. –3 d. 2

❑ THE DOMAIN OF A FUNCTION

The **domain** of a function is the set of all legal inputs.

We know that the ***domain*** of a function is the set of legal inputs to a function. Why is the domain a vital idea to consider? Total havoc can result when a number outside the domain is introduced into the function (like in a spreadsheet or a programming language).

For example, if our computer application tries to let $x = 0$ in the function $y = \frac{1}{x}$, we're sunk — the computer will stop execution of the program and give an error message (or worse, freeze up!). And if we allow $x = -4$ in the function $y = \sqrt{x}$, then we're really up a creek, since $\sqrt{-4}$ is not a real number, and we're assuming that real numbers are all we have at our disposal in this book.

To review, the **domain** of a typical function of x found in an algebra course is the set of all real numbers that are legal for x to be in the formula. For example, the domain of the function $y = x^2$ is $\mathbb{R}$, all the real numbers (since any real number can be squared without any serious problem). But in the formula $y = \frac{3}{x-7}$, x can be any real number except 7 (why?), and therefore the domain is $\mathbb{R} - \{7\}$, all the real numbers with 7 removed.

Homework

9. Consider the function $y = \sqrt{2x-8}$.
   Calculate the functional value for each x:
   a. 12 b. 36 c. 4.5 d. 4 e. 3

   From these results, we see that the values 12, 36, 4.5, and 4 are in the **function's domai**n, while 3 is not.

10. Let h be the function defined by $y = \frac{1}{x^2-9}$. Calculate each functional value:

x	-4	-3	-2	0	1	3	4	5
y								

   From this table of inputs and outputs, guess what the function's domain is.

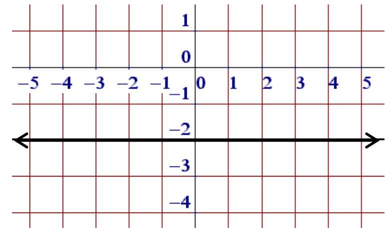
11. Consider the function given by the formula $y = x^3 - x^2$. What is the domain of g?

12. What is the domain of the function $y = \frac{2}{x+3}$?

❑ *FINDING THE DOMAIN*

I. $y = -2$

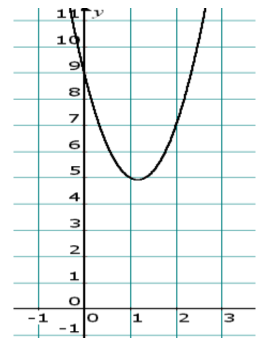
In this equation, the x isn't even mentioned. There can, therefore, be no restrictions on x. That is, x can be any real number. Thus, the domain is



$$\boxed{\mathbb{R}}$$

II. $y = 3x^2 - 7x + 9$

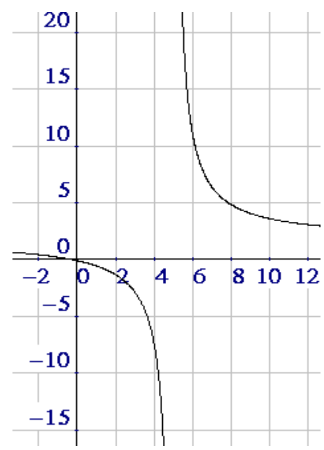
We ask ourselves: What are the legal x's? Well, x can be anything, since the operations in the formula could not possibly be a cause for concern. The domain is therefore



$$\boxed{\mathbb{R}}$$

III. $y = \dfrac{7x+2}{4x-20}$

Now we've got something interesting to look at. Question: What can go wrong in a division problem? Answer: The possibility of dividing by zero. We must ensure that x is <u>never</u> allowed to be a number that would make the denominator zero. So we find out what value(s) of x <u>would</u> make the bottom zero, and then <u>don't allow those x's to be in the domain</u>! Setting the bottom to zero gives



$$4x - 20 = 0 \longleftarrow$$ This is what we <u>don't</u> want to happen.

$$\Rightarrow \quad 4x = 20$$

$$\Rightarrow \quad x = 5$$

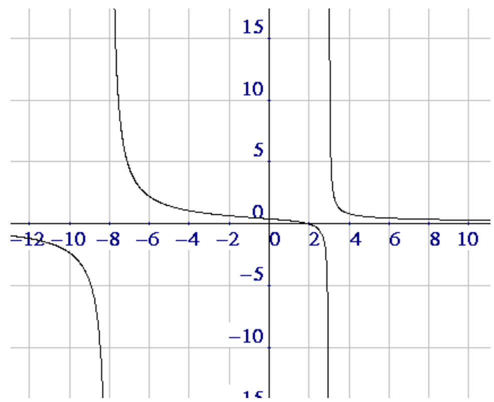
Thus, if $x = 5$, the denominator is zero, which is absolutely forbidden! So, the domain of this function is the set of <u>all</u> real numbers <u>except</u> 5:

$$\mathbb{R} - \{5\}$$

IV. $y = \dfrac{5x - 10}{x^2 + 5x - 24}$

As in the previous example, we must make certain that the denominator is <u>never</u> zero; let's see what values of x make it zero, and then <u>exclude</u> such values from our domain:

$$x^2 + 5x - 24 = 0$$
$$\Rightarrow (x + 8)(x - 3) = 0$$
$$\Rightarrow x + 8 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow x = -8 \text{ or } x = 3$$



We conclude that the domain is all real numbers <u>except</u> −8 and 3:

$$\mathbb{R} - \{-8, 3\}$$

The Quadratic Formula could have been used to solve the quadratic equation in this problem.

V. $y = \dfrac{2x - 8}{x^2 + 10x + 25}$

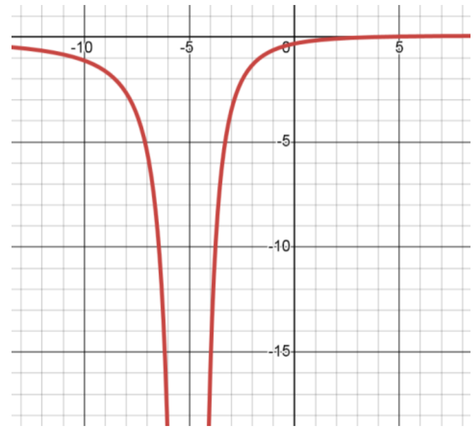
First, we note that it's perfectly O.K. for the numerator of a fraction to be 0, so the numerator has no role in determining the function's domain. We focus on determining what values of x would make the <u>bottom</u> zero, and then <u>exclude</u> those values from the domain:

$$x^2 + 10x + 25 = 0$$

$$\Rightarrow \quad (x+5)(x+5) = 0$$

$$\Rightarrow \quad x+5 = 0 \text{ or } x+5 = 0$$

$$\Rightarrow \quad x = -5 \text{ or } x = -5$$



We see that the only value of x that is <u>not</u> allowed in the domain is $x = -5$. Hence, the domain of the function is all real numbers except –5:

$$\boxed{\mathbb{R} - \{-5\}}$$

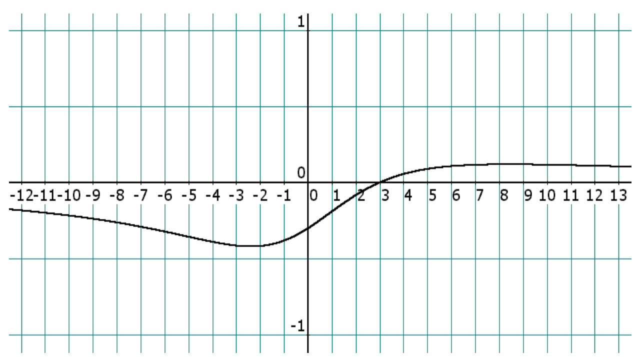
VI. $y = \dfrac{2x-6}{x^2+20}$

Let's see what makes the denominator zero (so we can <u>exclude</u> it from the domain.)

$$x^2 + 20 = 0$$

$$\Rightarrow \quad x^2 = -20$$

$\Rightarrow \quad x = \pm\sqrt{-20}$, which are <u>not</u> real numbers.



What do we conclude here? Well, we tried to figure out what values of x would make the denominator zero, so that we could exclude such an x from being in the domain. But there aren't any values of x that make the denominator zero. So there is nothing to exclude from the domain. Therefore, <u>every</u> real number is allowed in the function. The domain is

$$\boxed{\mathbb{R}}$$

Homework

Find the domain of each function:

13. $y = \pi$

14. $y = \dfrac{x^2 - 9}{9x - 7}$

15. $y = \dfrac{2x + 1}{x^2 - 100}$

16. $y = \dfrac{x^2 - 25}{x^2 + 49}$

17. $y = \sqrt{2} + \sqrt{3}$

18. $y = \sqrt{3}$

19. $y = \dfrac{2x - 3}{2x^2 + 3x}$

20. $y = \dfrac{3}{2x - x^2}$

21. $y = x^3 - 8$

22. $y = \dfrac{9x - 7}{2x + 9}$

23. $y = \dfrac{5x + 25}{x^2 - 144}$

24. $y = \dfrac{x}{x^2 + 3}$

25. $y = \sqrt{2}\,x + 9$

26. $y = \dfrac{x - 2}{x^2 - 81}$

27. $y = \dfrac{3}{2x^2 - 5x - 3}$

28. $y = 9$

29. $y = \dfrac{1}{x^2 + 10x + 25}$

30. $y = \dfrac{2x - 7}{8x^2 - 10x - 3}$

31. $y = \dfrac{x + 10}{x^2 + 100}$

32. $y = \dfrac{x^2 + 5x + 1}{4x^2 - x - 3}$

33. $y = \dfrac{3x + 7}{10x^2 - 20x}$

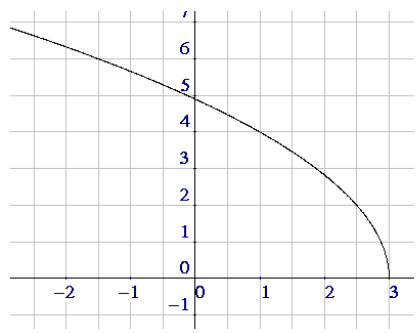
34. $y = \dfrac{x^2 - 14}{14x^2 - 42x}$

❑ THE DOMAIN OF SQUARE ROOT FUNCTIONS

EXAMPLE 1: **Find the domain of $y = \sqrt{24 - 8x}$.**

Solution: In this formula, we ask ourselves if anything could possibly go wrong using certain values of x. Well, the square root of a negative number doesn't exist in the real numbers, so certainly something could go wrong — for instance, if $x = 4$, then we'd have $y = \sqrt{-8}$.

Now we need a statement that describes the legal x's. How about this:



The square root of a quantity is defined (as a real number) only when that quantity is greater than or equal to zero.

So, in this example, the radicand, $24 - 8x$, must be greater than or equal to 0; i.e., **≥ 0**

$$24 - 8x \ge 0 \quad \text{(the radicand must be 0 or positive)}$$

$$\Rightarrow \quad -8x \ge -24 \quad \text{(subtract 24 from each side)}$$

$$\Rightarrow \quad x \le 3 \quad \text{(divide by –8, and reverse the inequality)}$$

Therefore, the domain is the set of all real numbers which are less than or equal to 3:

$\boxed{x \le 3}$ or, **$(-\infty, 3]$** in interval notation

Homework

Find the domain of each function:

35. $y = \sqrt{3x+12}$

36. $y = \sqrt{8-2x}$

37. $y = \sqrt{7x-15}$

38. $y = \sqrt{-4x+1}$

39. $y = \sqrt{17-17x}$

40. $y = \sqrt{4x+12}$

41. $y = \sqrt{4x+1}$

42. $y = \sqrt{10-10x}$

43. $y = \sqrt{20x}$

44. $y = \sqrt{20-x}$

45. $y = \sqrt{-3x-15}$

46. $y = \sqrt{-30-7x}$

Review Problems

Find the domain of each function:

47. $y = \sqrt{2}$

48. $y = \dfrac{2x-6}{x^2+4x+3}$

49. $f(x) = \pi$

50. $y = \pi x^3 - 17x + 1$

51. $y = \dfrac{x+2}{x-3}$

52. $y = \dfrac{3}{x^2-x-20}$

53. $y = \dfrac{20}{x^2-x}$

54. $y = \dfrac{\pi}{13x^2+26x}$

55. $y = \sqrt{18-9x}$

56. $y = \dfrac{4-x}{x^2-400}$

57. $y = \dfrac{13x-13}{9x^2-42x+49}$

58. $y = \dfrac{17x-17}{25x^2+20x+4}$

59. $y = \dfrac{x^2-5x+6}{50}$

60. $y = \dfrac{\sqrt{2}}{3}$

61. $y = \sqrt{12x-16}$

62. $y = \sqrt{10-5x}$

63. $y = \dfrac{x^2-49}{2x+17}$

64. $y = \dfrac{\sqrt{3}}{9x^2-49}$

Solutions

1. a. 33/2 b. 105/4 c. 21 d. Undefined
e. Undefined <u>Domain</u>

2. [3, 5]

3. $\mathbb{R}$

4. All four numbers are legal inputs, since all four numbers could be multiplied by 2 and then have 10 added on with no problem.

5. 3 is a legal input, since $\dfrac{3}{3-2} = \dfrac{3}{1} = 3$; no muss, no fuss.

2 is <u>not</u> legal, since $\dfrac{3}{2-2} = \dfrac{3}{0} = \textit{Undefined}$.

π is legal because nothing can go wrong.

0 is legal because $\dfrac{3}{-2}$ is a perfectly fine number.

6. $\sqrt{7}$ is a perfectly fine real number, so 7 is a legal input.
$\sqrt{0}$ is also a real number (= 0), so 0 is legal.
$\sqrt{-9}$ is <u>not</u> a real number, so this number doesn't exist in our current world; hence, –9 is <u>not</u> a legal input.
$\sqrt{99}$ is some real number (9 point something), so 99 is legal.

7. When $x = 12$, we get $12^2 - 144 = 144 - 144 = 0$ in the denominator. Zero in the denominator? I don't think so! Thus, 12 is <u>not</u> a legal input. 20 is O.K. to use for x, just as 0 is fine for x. So 20 and 0 are legal inputs. But when $x = -12$, we're in the same boat as when x was 12. This is because when $x = -12$ we get $144 - 144 = 0$ in the denominator, which is certainly not allowed. Therefore, –12 is <u>not</u> legal.

8. All four values of x are legal, since none of them makes the denominator zero.

9. a. 4 b. 8 c. 1 d. 0 e. Undefined

10. 1/7; Undefined; –1/5; –1/9; –1/8; Undefined; 1/7; 1/16
When $x = 3$ or -3, the function is undefined due to dividing by zero. It seems that no other inputs would produce a zero in the denominator. So, our guess would be that the domain of h is all real numbers <u>except</u> 3 and –3. This domain can be written $\mathbb{R} - \{3, -3\}$, or $\mathbb{R} - \{\pm 3\}$

11. $\mathbb{R}$ **12**. $\mathbb{R} - \{-3\}$ **13**. $\mathbb{R}$ **14**. $\mathbb{R} - \left\{\frac{7}{9}\right\}$ **15**. $\mathbb{R} - \{\pm 10\}$

16. $\mathbb{R}$, and here's why: No matter what x is, x^2 is at least 0 (since x^2 is never negative). Now add 49 to something that is at least 0, and you now have a number which is at least 49; so the denominator $x^2 + 49$ can <u>never</u> be 0. Since dividing by 0 is the only critical issue in this function, the domain is all real numbers.

17. $\mathbb{R}$ **18**. $\mathbb{R}$ **19**. $\mathbb{R} - \left\{0, -\frac{3}{2}\right\}$ **20**. $\mathbb{R} - \{0, 2\}$

21. $\mathbb{R}$ **22**. $\mathbb{R} - \left\{-\frac{9}{2}\right\}$ **23**. $\mathbb{R} - \{\pm 12\}$ **24**. $\mathbb{R}$

25. $\mathbb{R}$

26. $\mathbb{R} - \{\pm 9\}$

27. $\mathbb{R} - \left\{3, -\frac{1}{2}\right\}$

28. $\mathbb{R}$

29. $\mathbb{R} - \{-5\}$

30. $\mathbb{R} - \left\{-\frac{1}{4}, \frac{3}{2}\right\}$

31. $\mathbb{R}$

32. $\mathbb{R} - \left\{1, -\frac{3}{4}\right\}$

33. $\mathbb{R} - \{0, 2\}$

34. $\mathbb{R} - \{0, 3\}$

35. $x \geq -4$

36. $x \leq 4$

37. $x \geq \frac{15}{7}$

38. $x \leq \frac{1}{4}$

39. $x \leq 1$

40. $x \geq -3$

41. $x \geq -\frac{1}{4}$

42. $x \leq 1$

43. $x \geq 0$

44. $x \leq 20$

45. $x \leq -5$

46. $x \leq -\frac{30}{7}$

47. $\mathbb{R}$

48. $\mathbb{R} - \{-1, -3\}$

49. $\mathbb{R}$

50. $\mathbb{R}$

51. $\mathbb{R} - \{3\}$

52. $\mathbb{R} - \{5, -4\}$

53. $\mathbb{R} - \{0, 1\}$

54. $\mathbb{R} - \{0, -2\}$

55. $x \leq 2$

56. $\mathbb{R} - \{\pm 20\}$

57. $\mathbb{R} - \left\{\frac{7}{3}\right\}$

58. $\mathbb{R} - \left\{-\frac{2}{5}\right\}$

59. $\mathbb{R}$

60. $\mathbb{R}$

61. $x \geq \frac{4}{3}$

62. $x \leq 2$

63. $\mathbb{R} - \left\{-\frac{17}{2}\right\}$

64. $\mathbb{R} - \left\{\pm \frac{7}{3}\right\}$

"It is the Law that any difficulties that can come to you at any time, no matter what they are, must be exactly what you need most at the moment, to enable you to take the next step forward by overcoming them. The only real misfortune, the only real tragedy, comes when we suffer without learning the lesson."

– Emmet Fox